

Strictly Positive Definite Functions

Kuei-Fang Chang

*Department of Applied Mathematics, Feng Chia University,
Tai-Chung Taiwan, People's Republic of China*

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We give a complete characterization of the strictly positive definite functions on the real line. By Bochner's theorem, this is equivalent to proving that if the separated sequence of real numbers $\{a_n\}$ describes the points of discontinuity of a distribution function, there exists an almost periodic polynomial with the zeros $\{a_n\}$. We prove a useful necessary condition that every strictly normalized, positive definite function f satisfies $|f(x)| < 1$ for all $x \neq 0$. It is a sufficient condition for strictly positive definiteness that if the carrier of a nonzero finite Borel measure on \mathbb{R} is not a discrete set, then the Fourier–Stieltjes transform $\hat{\mu}$ of μ is strictly positive definite. © 1996 Academic Press, Inc.

1. INTRODUCTION

From the standpoint of interpolation theory, the positive definiteness (see [13]) of a function f is not strong enough; it allows us to conclude only that a matrix $f(x_i - x_j)$ is nonnegative definite. This matrix may therefore be singular. We must require our functions to be strictly positive definite, so that the corresponding matrices will be positive definite. For this purpose we first introduce the definition of a strictly positive definite function on the real field \mathbb{R} .

DEFINITION 1.1. Let f be a complex valued continuous function defined on \mathbb{R} . Then f is said to be

(i) *positive definite* on \mathbb{R} if the n by n matrix $[f(x_i - x_j)]$ is nonnegative definite for all choices of points $\{x_1, \dots, x_n\} \subset \mathbb{R}$ and all $n = 1, 2, \dots$.

(ii) *strictly positive definite* on \mathbb{R} if the n by n matrix $[f(x_i - x_j)]$ is positive definite for all choices of pairwise distinct points $\{x_1, \dots, x_n\} \subset \mathbb{R}$ and all $n = 1, 2, \dots$.

We have the following elementary property for a positive definite function (see, [9, p. 161]).

THEOREM 1.2. *Let f be positive definite on \mathbb{R} . Then*

- (i) $f(0) \geq 0$.
- (ii) $f(-x) = \overline{f(x)}$ for all real x .
- (iii) $|f(x)| \leq f(0)$ for all real x .

If a positive definite function f satisfies $f(0) = 0$, then f is the zero function. We say that a positive definite function f is *normalized* if $f(0) = 1$. We define the *characteristic function* f of a distribution function F (see [11, p. 2]) by the generalized Fourier transform of F

$$f(x) = \int_{\mathbb{R}} e^{ixy} dF(y), \quad x \in \mathbb{R}.$$

Bochner [3] showed the following famous result between characteristic functions and positive definite functions.

THEOREM 1.3. *Let f be a complex-valued function defined on \mathbb{R} . Then f is a normalized, positive definite function if and only if f is a characteristic function of a distribution function.*

Recall [11, p. 4] that every distribution function F can be decomposed uniquely into three parts

$$F(x) = \alpha_1 F_d(x) + \alpha_2 F_{ac}(x) + \alpha_3 F_s(x). \quad (1.1)$$

Here F_d , F_{ac} , and F_s are three distribution functions. The functions F_{ac} and F_s are both continuous; however, F_{ac} is absolutely continuous while F_s is singular and F_d is a step function. The coefficients α_i are nonnegative and $\sum_{k=1}^3 \alpha_k = 1$. The distribution functions F_d , F_{ac} , and F_s are called the discrete, the absolutely continuous, and the singular parts, respectively, of $F(x)$. It is well known [11, p. 36] that a distribution function is discrete if and only if its characteristic function is almost periodic. In order to realize strictly normalized, positive definite functions, one almost inevitably encounters all three kinds of distributions when applying Bochner's theorem.

This paper is organized as follows. In section two, we prove a useful necessary condition for every normalized, strictly positive definite function to satisfy $|f(x)| < 1$ for all $x \neq 0$, and a sufficient condition for strictly positive definiteness: if the carrier of a nonzero finite Borel measure on \mathbb{R} is not a discrete set, then the Fourier–Stieltjes transform $\hat{\mu}$ of μ is strictly

positive definite. The last section is devoted to necessary and sufficient condition for strictly positive definiteness.

2. NECESSARY OR SUFFICIENT CONDITIONS

A discrete distribution is a *lattice distribution* [11, p. 17] if its points of discontinuity are of the form $a + kd$, where a, d are constant ($d > 0$) and k is an integer. We know that [11, p. 18] a characteristic function f is the characteristic function of a lattice distribution if and only if there exists a nonzero real number x_0 such that $|f(x_0)| = 1$. If, in particular, $|f(x)| = 1$ for all x , then f is the characteristic function of a degenerate distribution. We give a useful necessary condition for strictly positive definiteness but this is elementary. We leave the proof to the reader as a calculus exercise.

THEOREM 2.1. *A strictly normalized, positive definite function f satisfies $|f(x)| < 1$ for all $x \neq 0$.*

In the next theorem, we use the *carrier* of a Borel measure μ on \mathbb{R} . This is defined to be the set

$$\Omega = \mathbb{R} \setminus \bigcup \{O : O \text{ is open and } \mu(O) = 0\}.$$

This concept is discussed in [12, p. 308]. It is obvious that Ω is closed, and $\mu(\mathbb{R} \setminus \Omega) = 0$.

THEOREM 2.2. *Let μ be a nonzero, finite, Borel measure on \mathbb{R} such that the carrier of μ is not a discrete set. Then the generalized Fourier transform $\hat{\mu}$ of μ is strictly positive definite on \mathbb{R} .*

Proof. In order to prove that $\hat{\mu}$ is strictly positive definite, let x_1, x_2, \dots, x_n be distinct points in \mathbb{R} and let c_1, c_2, \dots, c_n be complex numbers, not all zero. Then

$$\begin{aligned} \sum_k \sum_j \bar{c}_k c_j \hat{\mu}(x_k - x_j) &= \sum_k \sum_j \bar{c}_k c_j \int_{\mathbb{R}} e^{-i(x_k - x_j)y} d\mu(y) \\ &= \int_{\mathbb{R}} \left(\sum_k \bar{c}_k e^{-ix_k y} \right) \left(\sum_j c_j e^{ix_j y} \right) d\mu(y) \\ &= \int_{\mathbb{R}} \left| \sum_j c_j e^{ix_j y} \right|^2 d\mu(y) \\ &= \int_{\mathbb{R}} |g(y)|^2 d\mu(y), \end{aligned}$$

where we have set $g(y) = \sum_j c_j \exp(ix_j y)$. Suppose that the final integral is zero. For each positive integer m , let h_m be a continuous function such that $h_m(x) = 1$ when $|x| \leq m$ and $h_m(x) = 0$ when $|x| \geq m + 1$. Then gh_m is a continuous function having compact support. Also we have

$$\int_{\mathbb{R}} |g(x)|^2 h_m(x) d\mu(x) = 0.$$

By [12, p. 308], $g(x)h_m(x) = 0$ for all x in the carrier of μ . It follows that $g(x) = 0$ on the carrier of μ . Hence the carrier of μ is a subset of the zero set of g . The latter is discrete since g can be regarded as an entire function of exponential type on \mathbb{C} . This is a contradiction to our hypotheses. Hence, $\int_{\mathbb{R}} |g|^2 d\mu > 0$. ■

COROLLARY 2.3. *Let f be a nonnegative Borel measurable function on \mathbb{R} . If f satisfies $0 < \int_{\mathbb{R}} f < \infty$, then the Fourier transform \hat{f} of f is strictly positive definite.*

Proof. We reduce this to Theorem 2.2 by using the measure μ defined (for any Borel set) by the equation

$$\mu(A) = \int_A f(x) dx.$$

The carrier of μ is

$$\begin{aligned} \text{car}(\mu) &= \mathbb{R} \setminus \bigcup \{ O : O \text{ open, } O \subset Z(f) \} \\ &= \bigcap \{ K : K \text{ closed, } \mathbb{R} \setminus K \subset Z(f) \} \\ &= \bigcap \{ K : K \text{ closed, } K \supset \mathbb{R} \setminus Z(f) \} \\ &= \overline{\{ x : F(x) \neq 0 \}} \\ &= \text{supp}(f). \end{aligned}$$

Since $f \neq 0$, its support is a set of positive Lebesgue measure. Hence $\text{car}(\mu)$ is not discrete. ■

In particular, if f is a probability density function, we have the following.

COROLLARY 2.4. *Let f be a characteristic function corresponding to an absolutely continuous distribution. Then the derivative F' of F is a probability density function and f is a strictly positive definite function on \mathbb{R} .*

Assume that f is a characteristic function corresponding to a distribution function F and we use the same notation as in the proof of Theorem 2.2.

Let $\{a_k\}$ be a sequence which contains all points of discontinuity of F . Then the discrete distribution function F_d is given by

$$F_d(x) = \sum_k p_k \varepsilon(x - a_k).$$

Here we denote the distribution function ε by $\varepsilon(x) = 0$ for $x < 0$ and $\varepsilon(x) = 1$ for $x \geq 0$. The p_k satisfy the relations $p_k > 0$, and $\sum_k p_k = 1$. In order that f is strictly positive definite, it suffices that by Eq. (1.1) the integrals

$$\begin{aligned} \int_{\mathbb{R}} |g(y)|^2 dF(y) &= \int_{\mathbb{R}} |g(y)|^2 d(\alpha_1 F_d(y) + \alpha_2 F_{ac}(y) + \alpha_3 F_s(y)) \\ &= \alpha_1 \int_{\mathbb{R}} |g(y)|^2 dF_d(y) + \alpha_2 \int_{\mathbb{R}} |g(y)|^2 dF_{ac}(y) \\ &= \alpha_1 \sum_k p_k |g(a_k)|^2 + \alpha_2 \int_{\mathbb{R}} |g(y)|^2 dF_{ac}(y) \end{aligned}$$

are positive, since $F'_s = 0$ almost everywhere (see [7, p. 337]). Without loss of generality, we can assume either $\alpha_1 = 1$ or $\alpha_2 = 1$. If $\alpha_2 = 1$, then by the Radon–Nikodym theorem and Theorem 2.3, f is strictly positive definite. On the other hand, $\int_{\mathbb{R}} |g(y)|^2 dF_d(y) = \sum_k p_k |g(a_k)|^2$ and $\sum_k p_k |g(a_k)|^2 = 0$ if and only if $g(a_k) = 0$ for all k . This will be the main tool for the final characterization of strictly positive definite functions. We denote a special class of almost periodic polynomials by

$$\mathcal{P} = \left\{ g(x) = \sum_{k=1}^n c_k e^{ix_k x} : c_k \in \mathbb{C}, x_k \in \mathbb{R}, \text{ and } n \in \mathbb{N} \right\},$$

where the c_k are not all zero and the x_k are pairwise distinct. Next, we will describe the zero structure of functions $g \in \mathcal{P}$. This is related to a large class of entire functions of exponential type whose zeros $\{x_n\}$ give rise to bases $\{e^{ix_n \cdot}\}$ of complex exponentials.

DEFINITION 2.5. A entire function f of exponential type α is of sine type if there exist positive constants A , B , and H such that

$$Ae^{\alpha|y|} \leq |f(x + iy)| \leq Be^{\alpha|y|},$$

where x and y are real and $|y| \geq H$.

According to the definition, a function of sine type is bounded on \mathbb{R} . The zeros are simple and lie in a strip parallel to the real axis (see [14, p. 171]).

Golovin [6] proved the following. Let $\{x_n\}$ be the set of zeros of a sine-type function and let the width of its indicator diagram (see [2, p. 73]) be equal to a ($a > 0$). Then $\{e^{ix_n}\}$ form a Riesz basis in $L^2(I)$ and I is an interval with length a . Krein and Levin [10, p. 458] gave the following equivalent conditions for a sine-type function.

THEOREM 2.6. *Assume that $\{x_k\}$ is separated and $x_k = ck + \psi(k)$, where c is constant and $\{\psi(k)\}_k$ is almost periodic. Let the function g be defined by*

$$g(z) = \lim_{N \rightarrow \infty} \prod_{-N}^N \left(1 - \frac{z}{x_k}\right).$$

Then the following are equivalent:

- (i) g is a function of sine type.
- (ii) There exists an entire function of exponential type not exceeding π , bounded on the real axis, and taking values $(-1)^k \psi(k)$ for all integer k .
- (iii) For arbitrary integer τ and for any $h > 0$ the linear functional L_τ defined by

$$L_\tau[\psi] = \sum_{k \in \mathbb{Z}} [\psi(k + \tau) - \psi(k)] \frac{k}{k^2 + h^2}$$

is uniformly bounded in τ .

In the next section, we will show that every element of class \mathcal{P} is of sine type.

3. ALMOST PERIODIC FUNCTIONS

The class of almost periodic functions was initiated by Bohr [4] and developed by Besicovitch [1]. A function such as

$$g(x) = \sum_{k=1}^n c_k e^{ix_k x},$$

where c_k are complex numbers and x_k are real numbers, is called an *almost periodic polynomial*. A complex-valued function G defined on \mathbb{R} is called *almost periodic* if for any $\delta > 0$, there is an almost periodic polynomial g_δ such that

$$|G(x) - g_\delta(x)| < \delta \quad \text{for all } x \in \mathbb{R}.$$

We state the following well-known equivalence result (see [5]) for almost periodic functions.

THEOREM 3.1. *The following statements are equivalent:*

- (i) *G is an almost periodic function.*
- (ii) *From any sequence of the form $\{G(x+h_n)\}$, where h_n are real numbers, one can extract a subsequence converging uniformly on \mathbb{R} .*
- (iii) *For any $\delta > 0$, there exists a number $l(\delta) > 0$ with the property that any interval of length $l(\delta)$ of the real line contains at least one point with abscissa ζ , such that*

$$|G(x+\zeta) - G(x)| < \delta \quad \text{for all } x \in \mathbb{R}.$$

By Theorem 3.1, we can similarly define a complex-valued sequence to be *almost periodic* since such a sequence is regarded as a complex-valued function of an integer variable.

DEFINITION 3.2. A sequence $\{a_n\}$ is said to be almost periodic if for any $\delta > 0$ there corresponds an integer $N(\delta)$, such that among any N consecutive integers there exists an integer p with the property

$$|a_{n+p} - a_n| < \delta \quad \text{for all } n \in \mathbb{Z}.$$

The *mean value* of an almost periodic function G is defined by

$$M(G) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T+\alpha}^{T+\alpha} G(t) dt,$$

where the convergence is uniform relative to α and $M(G)$ is independent of α (see [5]). The *Fourier coefficients*

$$a(\lambda) = M(G(x) e^{-i\lambda x})$$

differ from zero only for a countable set of λ . This countable set $\{\lambda_n\}$ is called the *spectrum* of the function f or the set of its Fourier exponents, and the series

$$G(x) \sim \sum_n c_n e^{i\lambda_n x} \quad (c_n = a(\lambda_n))$$

is the Fourier series of the function G . An important property between almost periodic function and its Fourier series is stated by the following theorem (see [5]).

THEOREM 3.3. *If the Fourier series of an almost periodic function is uniformly convergent, then the sum of the series is the given function.*

The following lemma is taken from Levin [10, p. 270].

LEMMA 3.4. *In order that all the roots of an entire almost periodic function of exponential type be situated in some strip parallel to the real axis, it is necessary and sufficient that the upper and lower bounds of the spectrum enter into the spectrum.*

THEOREM 3.5. *Every almost periodic polynomial of class \mathcal{P} is of sine type.*

Proof. Let $g \in \mathcal{P}$ and let $g(x) = \sum_{k=1}^n c_k e^{ix_k x}$ with $x_1 < x_2 < \dots < x_n$. An elementary calculation shows that

$$M(e^{ix_j x}, e^{ix_k x}) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Therefore, the spectrum of g is the set $\{x_k : k = 1, \dots, n\}$ and, hence, the zeros of g lie in a strip between $y = x_1$ and $y = x_n$ by Lemma 3.4. This implies that g is of sine type. ■

The following special subclass of sine-type functions is due to Levin [10, p. 271].

DEFINITION 3.6. A entire function is a function of class $[\Delta]$ if it is almost periodic with its whole spectrum in an interval of length 2Δ , the end-points of the interval belonging to the sequence of its spectrum.

By the same argument as in the proof of Theorem 3.5, we have the following result.

THEOREM 3.7. *Let $g \in \mathcal{P}$ and let $g(x) = \sum_{k=1}^n c_k e^{ix_k x}$ with $x_1 < x_2 < \dots < x_n$. Then g is a function of the class $[\Delta]$ with $\Delta = (x_n - x_1)/2$.*

Krein and Levin gave the following complete characterization of the set of zeros of an almost periodic functions belonging to the class $[\Delta]$. In fact, this is a reformulation of Theorem 2.6 (see [10, p. 453]).

THEOREM 3.8. *In order for g to be an almost periodic function of the class $[\Delta]$, it is necessary and sufficient that the following conditions hold:*

(i) g can be expressed in the form

$$g(z) = c \lim_{N \rightarrow \infty} \prod_{-N}^N \left(1 - \frac{z}{x_k} \right) \quad (c \text{ a constant});$$

(ii) The roots $\{x_k\}$ are almost periodic and can be given by the formula

$$x_k = \frac{\pi}{\Delta} k + \psi(k),$$

where $\psi(k)$ is a complex-valued, bounded sequence.

(iii) For arbitrary integer τ and for any $h > 0$ the linear functional L_τ defined by

$$L_\tau[\psi] = \sum_{k \in \mathbb{Z}} [\psi(k + \tau) - \psi(k)] \frac{k}{k^2 + h^2}$$

is uniformly bounded in τ .

Remark. (1) It is well known [10, p 446] that if $\{x_n\}$ is the sequence of the roots of a function g of the class $[\Delta]$, enumerated in the order of increasing real parts, then

$$\operatorname{Re} x_k = \frac{\pi}{\Delta} k + \phi(k),$$

where ϕ is an almost periodic function. Since all the zeros of g lie in a strip parallel to the real axis, it follows from the above equation that condition (ii) of Theorem 3.8 holds.

(2) Conditions (iii) is independent of the remaining conditions of Theorem 3.8 (see [10, p. 455]).

THEOREM 3.9. *Let $\{a_k\}$ be a separated sequence in \mathbb{R} . In order that $g(x) = \sum_{k=1}^n c_k e^{ix_k x}$ with $x_1 < x_2 < \dots < x_n$ is an almost periodic polynomial of the class \mathcal{P} with the zeros $\{a_k\}$, it is necessary and sufficient that the following conditions hold:*

(i) g can be expressed in the form

$$g(z) = c \lim_{N \rightarrow \infty} \prod_{-N}^N \left(1 - \frac{z}{a_k} \right) \quad (c \text{ a constant}).$$

(ii) The roots $\{a_k\}$ form an almost periodic sequence and can be given by the formula

$$a_k = \frac{\pi}{\Delta} k + \phi(k),$$

where $\Delta = (x_n - x_1)/2$ and ϕ is an almost periodic function.

(iii) For arbitrary integer τ , the linear functionals $L_\tau[\phi]$ are uniformly bounded in τ , where

$$L_\tau[\phi] = \sum_{k \in \mathbb{Z}} [\phi(k + \tau) - \phi(k)] \frac{k}{k^2 + h^2},$$

$\tau \in \mathbb{Z}$, and $h > 0$.

(iv) The spectrum of g has distinct and finite Fourier exponents.

Proof. Combining Theorems 3.7 and 3.8, conditions (i)–(iii) of Theorem 3.9 hold if and only if the function g is an almost periodic function of the class $[\Delta]$. Theorem 3.3 implies that under the conditions (i)–(iii) of Theorem 3.9, condition (iv) of Theorem 3.9 is equivalent for g being an almost periodic function of class \mathcal{P} . ■

EXAMPLE 3.10. Let the sequence $a_k = k + \delta \operatorname{sgn}(k)$, $0 < \delta < 1/4$, be the points of discontinuity of a discrete distribution function F . The set of linear functionals $L_\tau[\phi]$ is not bounded for all integer τ . This implies that by Theorem 2.6, a_k cannot be the roots of any sine-type function and hence the characteristic function f of such a distribution function F is strictly positive definite on \mathbb{R} due to Theorem 3.9. Note that the family $\{e^{iak}\}$ forms a Riesz basis in $L^2(-\pi, \pi)$ (see [8, p. 217]).

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